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# Dynamical Determinants via Dynamical Conjugacies for Postcritically Finite Polynomials

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We give an analogue of Levin–Sodin–Yuditskii's study of the dynamical Ruelle determinants of hyperbolic rational maps in the case of subhyperbolic quadratic polynomials. Our main tool is to reduce to an expanding situation. We do so by applying a dynamical change of coordinates on the domains of a Markov partition constructed from the landing ray at the postcritical repelling orbit. We express the dynamical determinants $d_\beta(z) = \exp - \sum_{k \geqslant 1} \frac{z^k}{k} \sum_{w \in \mathrm{Fix}\, f^k} \frac{1}{((f_c^k)'(w))^\beta} \frac{1}{1 - \frac{1}{(f_c^k)'(w)}}$ $(\beta \in \mathbb{Z}_+)$ as the product of an (entire) determinant with an explicit expression involving the postcritical repelling orbit, thus explaining the poles in $d_\beta(z)$.

**KEY WORDS:** Subhyperbolic/periodic/(post)critically finite quadratic polynomial; dynamical Fredholm determinant; Ruelle transfer operator; Yoccoz puzzle.

## 1. INTRODUCTION

Let $f$ be a rational function on the Riemann sphere, with bounded Julia set $J$. We shall recall some classical results of Ruelle and more recent work of Levin–Sodin–Yuditskii which hold under a hyperbolicity assumption, before moving to our study of subhyperbolic polynomials.

Assume then that $J$ is real and that $f$ is *hyperbolic* on $J$, i.e., there exist constants $K > 0$, $\rho > 1$ such that $|(f^k)'(x)| \geqslant K \cdot \rho^k$ for all $x$ in $J$ and

Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthday.

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all nonnegative integers $k$. The thermodynamical formalism of hyperbolic quadratic polynomials is well understood by now (see, e.g., refs. 1–4), we shall only recall here results on Ruelle transfer operators which are connected to our study. Consider the transfer operator

$$\mathscr{L}\varphi(x) = \sum_{fy=x} \frac{\varphi(y)}{(f'(y))^2}, \tag{1.1}$$

acting on the space of functions $\varphi$ analytic in a critical point-free complex neighbourhood of the Cantor set $J$. Ruelle[1, 5] observed that the operator $\mathscr{L}$ acting on bounded holomorphic functions in a neighbourhood of $J$ is nuclear in the sense of Grothendieck, in particular $\mathscr{L}$ admits a Fredholm determinant $\det(1-z\mathscr{L})$, that Ruelle expressed as the following dynamical function:

$$\det(1-z\mathscr{L}) = \exp - \sum_{k\geqslant 1} \frac{z^k}{k} \sum_{w\in \mathrm{Fix}\, f^k} \frac{1}{((f^k)'(w))^2} \frac{1}{1-\frac{1}{(f^k)'(w)}}. \tag{1.2}$$

Here, $\mathrm{Fix}\, f^k = \{w \in \mathbb{C} \mid f^k(w) = w\}$ (in particular we include complex fixed points—which will be present in most subhyperbolic examples introduced later—but do *not* include $w = \infty$).

More recently, Levin, Sodin, and Yuditskii [ref. 6, (4.15), (5.4)] obtained remarkable explicit expressions for $\det(1-z\mathscr{L})$, which only involve the iterates of the critical points of the expanding rational map $f$ (with bounded real Julia set). In the special case of a quadratic polynomial $f_c(z) = z^2+c$ with $c < -2$, they find

$$\begin{aligned}\det(1-z\mathscr{L}) &= \exp - \sum_{k\geqslant 1} \frac{z^k}{k} \sum_{w\in \mathrm{Fix}\, f_c^k} \frac{1}{((f_c^k)'(w))^2} \frac{1}{1-\frac{1}{(f_c^k)'(w)}}\\ &= 1 + \sum_{k\geqslant 1} \frac{(z/2)^k}{f_c(0)\cdots f_c^k(0)}.\end{aligned} \tag{1.3}$$

Our aim here is to obtain analogous results for the *subhyperbolic (i.e., postcritically finite) case of quadratic polynomials*. Note that Levin *et al.* extended their results[7] to more general hyperbolic rational maps $f$ (than in ref. 6), but considering only transfer operators $\mathscr{L}_Q\varphi(x) = \sum_{fy=x} \varphi(y)\, Q(y)$ associated to a rational weight $Q$ with *no* pole on the Julia set. (In ref. 7, Section 5 the case $Q(y) = (f'(y))^2$ is considered, but the analysis does *not* cover subhyperbolic polynomials.)

Our first observation (see Section 4 for a proof) is that the second equality in (1.3) still holds formally for *all* quadratic polynomials:

**Theorem A (Formal Levin–Sodin–Yuditskii Identity).** Let $f_c$ be a quadratic polynomial. Then we have (in the sense of formal power series with coefficients rational functions of $c$):

$$\exp - \sum_{k \geqslant 1} \frac{z^k}{k} \sum_{w \in \operatorname{Fix} f_c^k} \frac{1}{((f_c^k)'(w))^2} \frac{1}{1 - \frac{1}{(f_c^k)'(w)}} = 1 + \sum_{k \geqslant 1} \frac{(z/2)^k}{f_c(0) \cdots f_c^k(0)}.$$

Note that if $f_c$ is a Collet–Eckmann quadratic polynomial, i.e.,

$$\liminf_{k \to \infty} \frac{1}{k} \log |(f_c^k)'(f_c(0))| > 0,$$

then the identity in Theorem A is an equality between convergent power series.

In the following we make the additional assumption that the critical point is preperiodic (many of our ideas could be extended to the Collet–Eckmann case). The thermodynamical formalism of preperiodic quadratic polynomials is not as straightforward as that of hyperbolic polynomials (see, e.g., ref. 8 for a pedestrian construction of the absolutely continuous invariant measure and refs. 9 and 10 for a more recent analysis of the phase transitions).

The finiteness of the postcritical orbit implies that the right-hand-side in Theorem A defines a rational function. Our aim is now to interpret the left-hand side in Theorem A as a (necessarily modified, since it is not entire) dynamical determinant, similarly to the first equality of (1.3). For this, we shall conjugate our complex subhyperbolic quadratic polynomial with an expanding analytic dynamical system. This is related to the "Thurston orbifold" metric that Douady and Hubbard[(11)] construct in a ramified covering space over a neighbourhood of the Julia set in order to exhibit the hyperbolic properties of subhyperbolic polynomials (see comments after the proof of Theorem B). Here we shall use a more direct construction (as was that of ref. 8, on the real axis) in order to analyse the dynamical determinant.

More precisely, we first build a finite Markov partition in the complex plane reminiscent of (but not the same as) the Yoccoz puzzle (we consider external rays at the landing periodic point and iterate them backwards finitely many times). We then introduce conjugacies defined by iterates of the map itself, defining (Section 2) new coordinates for our dynamics. (This in fact is analogous to the conjugacy appearing in the towers of Young,[(12)] e.g.) We show in Section 3 that we thus obtain an expanding analytic system where the results of Ruelle[(5)] apply and give rise to a Ruelle–Grothendieck–Fredholm dynamical determinant. One needs to relate the

Fredholm determinants of the system in the old and the new coordinates, this is Lemma 6 in Section 3 where we prove that the only special periodic orbit is the postcritical one. Our main result is the following theorem, proved in Section 4:

**Theorem B (Determinant for Subhyperbolic Quadratic Polynomials).** Let $f_c$ be a quadratic polynomial with a preperiodic critical point $c$, landing on a repelling periodic point $c_n$ of period $m \geqslant 1$, multiplier $\lambda_f$ of modulus $> 1$, and admitting $q \geqslant 1$ external rays, each of period $d \cdot m$. For every nonnegative integer $\beta$, and any function $h$ holomorphic in a neighbourhood of the Julia set of $f_c$, define

$$d_\beta(z) = \exp - \sum_{k \geqslant 1} \frac{z^k}{k} \sum_{w \in \mathrm{Fix}\, f^k} \frac{\prod_{\ell=0}^{k-1} h(f_c^\ell(w))}{((f_c^k)'(w))^\beta} \frac{1}{1 - \frac{1}{(f_c^k)'(w)}}. \tag{1.4}$$

Then there is a choice of $\mu_f = \sqrt{\lambda_f^d}$ with $\mu_f^2 = \lambda_f^d$ such that, writing $h_m := \prod_{\ell=0}^{m-1} h(f_c^\ell(c_n))$, the product

$$\hat{d}_\beta(z) = d_\beta(z) \cdot \prod_{j \geqslant 0} \frac{(1 - z^{dm} h_m^d \mu_f^{-j-\beta})^q}{1 - z^m h_m \lambda_f^{-j-\beta}} \tag{1.5}$$

extends to an entire function. The zeroes of this function are exactly the inverses of the nonzero eigenvalues of the transfer operator

$$\mathscr{L}_\beta \varphi(x) = \sum_{f_c(y) = x} \frac{h(y) \cdot \varphi(y)}{(f_c'(y))^\beta}$$

acting on a Banach space $\mathscr{B}_\beta$ of multivalued functions (with prescribed singularities along the postcritical orbit) in a neighbourhood of the Julia set of $f_c$.

The definition of $\sqrt{\lambda_f^d}$ is given in (3.12), and $\mathscr{B}_\beta$ is constructed in the proof of Theorem B in Section 4 (see (4.6), and our comments there linking our construction with the ramified covering in ref. 11).

The simplest illustration of Theorem B is given by the quadratic map $f_{-2}(z) = z^2 - 2$, where $f_{-2}^2(0) = 2$ is a fixed point with derivative $\lambda_f = +4$. We may compute $d_\beta(z)$ for $\beta = 2$ from Theorem A, and find $1 - \sum_{k \geqslant 1} z^k / 4^k = (1 - z/2)/(1 - z/4)$. Theorem B (here $m = d = q = 1$, and (3.12) gives $\sqrt{4} = +2$) then says that $\hat{d}_2(z) = (1 - z/2) \prod_{j \geqslant 0} (1 - z/(8 \cdot 2^{2j}))$, an entire function which vanishes at $z = 2$ and $z = 8 \cdot 2^{2j}$ for all nonnegative integer $j$.

In fact, the proof of Theorem B is local, and we shall prove the following generalisation in Section 4 (see, e.g., ref. 13 for the definition of a quadratic-like map):

**Theorem B′ (Determinant for Subhyperbolic Quadratic-Like Maps).** The statement of Theorem B also holds if $f_c$ is a quadratic-like map with a preperiodic critical point landing on a repelling periodic point.

We mention that Epstein[14] and Ushiki[15, 16] have independently studied related determinants with different approaches. The tower construction in Smirnov's Ph.D. [ref. 10, Section 4.1] is more closely related to our method, but was used towards different goals (in particular no determinants were considered).

Finally, we would like to point out that the assumption that all critical points are mapped to repelling periodic orbits is enough to extend our results to polynomials (or polynomial-like maps) of higher degree, or even to a suitable class of rational maps. (The square root appearing in Theorem B should be replaced by a root of degree of the appropriate critical point, and the $q$, $d$-counting becomes more complicated.)

## 2. DYNAMICAL CHARTS FOR PREPERIODIC QUADRATIC POLYNOMIALS

Consider a quadratic polynomial $f(z)=z^2+c$ (we shall discuss quadratic-like maps only when proving Theorem B′ at the end of Section 4). Then $\infty$ is a super-attractive *critical fixed point* of $f$ with $f(\infty)=f^{-1}(\infty) =\infty$. Thus there are an open neighbourhood $V$ of $\infty$ and a unique holomorphic diffeomorphism $\phi$ defined on $V$, such that $\phi(\infty)=\infty$, $\phi'(\infty)=1$ and $\phi(z^2)=f(\phi(z))$. (This is the Boettcher conjugacy.) The other critical point of $f$ is 0. The set $C=\{c_j=f^j(0)\}_{j=0}^{\infty}$ is called the *critical orbit* of $f$. If $C$ does not intersect $V$, i.e., if it is bounded (the corresponding parameter value $c=c_1$ then lies in the Mandelbrot set, by definition), then $\phi$ can be holomorphically extended to the domain $\{z\in\overline{\mathbb{C}} \mid |z|>1\}$.

Henceforth, we assume that $C$ is bounded.

The image $B=\phi(\{z\in\overline{\mathbb{C}} \mid |z|>1\})$ is the basin of $\infty$ under the iteration of $f$. Its complement $K=\overline{\mathbb{C}}\setminus B$ is called the filled-in Julia set of $f$, and the boundary $J=\partial K$ is called the *Julia set* of $f$. (It is well known that $K$ is connected if and only if $C$ is bounded.)

The image of a circle $\{z\in\mathbb{C} \mid |z|=r\}$ of radius $r>1$ under $\phi$ is called an *equipotential curve* of $f$. The image $\gamma_\theta$ of a half-line $e_\theta=\{re^{2\pi i\theta} \mid r>1\}$ is called an *external ray of angle* $\theta$ of $f$. An external ray $\gamma_\theta$ is said to *land at a point* $w$ in $K$ if the limit $\lim_{r\to 1+}\phi(re^{2\pi i\theta})$ exists and is $w$.

A point in $K$ is called a *periodic point* of period $m>0$ if $f^j(p) \neq p$ for $1 \leqslant j < m$ and $f^m(p)=p$. For a periodic point $p$ of period $m$, its *multiplier* is $\lambda(p)=(f^m)'(p)$. The periodic point $p$ is called *repelling* if $|\lambda(p)|>1$, *attracting* if $|\lambda(p)|<1$, *neutral* if $|\lambda(p)|=1$. The following theorem is due to Douady (the polynomial need not be assumed quadratic):

**Douady Landing Theorem.** Let $f$ be a polynomial with a connected filled-in Julia set and let $p$ be a repelling periodic point of $f$ of period $m$. Then there is a nonempty finite set of (say $q \geqslant 1$) external rays landing at $p$. There is a divisor $d \geqslant 1$ of $q$ such that each of these landing rays is periodic of period $m \cdot d$.

A proof of the Douady landing theorem can be found, e.g., in ref. 17, Theorem I.A; ref. 18, Corollary B.1 (see ref. 19 for a statement without the assumption that the Julia set is connected).

Since $C$ is a finite set, then either 0 itself is periodic, or there is a nonzero periodic point in $C$. If 0 is not periodic but preperiodic (i.e., one of its forward iterates is periodic), then the polynomial $f$ is called *subhyperbolic* (or preperiodic, or (post)critically finite). (The corresponding parameter value $c$ is often called a *Misiurewicz point,* see, e.g., ref. 20 for some of their nontrivial properties.) In this case, it is known that the filled-in Julia set $K$ of $f$ has no interior and coincides with the Julia set $J$. Moreover, the periodic orbit in $C$ is repelling (see refs. 21 and 22). We shall restrict henceforth to subhyperbolic quadratic polynomials.

For a preperiodic quadratic polynomial $f$ we have

$$C=\{c_0=0, c_1=c,\dots, c_{n-1}, c_n,\dots, c_{n+m-1}, c_{n+m}=c_n\}, \tag{2.1}$$

with $n=\min\{j \geqslant 0, c_j \text{ is periodic }\} \geqslant 1$, and $\{c_n=f(c_{n+m-1}),\dots, c_{n+m-1}\}$ is a (repelling) periodic orbit, of multiplier denoted by

$$\lambda_f=(f^m)'(c_n). \tag{2.2}$$

Finally, we write

$$f(C)=\{c_1,\dots, c_{n-1}, c_n,\dots, c_{n+m-1}\}$$

for the postcritical orbit of $f$.

The Douady landing Theorem says that there is a finite number $q$ of external rays landing at $c_n$. Denote the union of the corresponding curves as $\Gamma_n$. Then $\Gamma_n$ can be pulled back by $f^{-1}$, giving a union $\Gamma_{n+m-1}$ of external rays landing at $c_{n+m-1}$ and $\Gamma_{n-1}$ of external rays landing at $c_{n-1}$. Through pull-back by appropriate branches of $f^{-1}$ we obtain a collection of external rays $\Gamma_j$ landing at $c_j$, for all $0 \leqslant j < n+m$. Note that $\#\Gamma_j = q$ for all $j \neq 0$ while $\#\Gamma_0 = 2q$ (the doubling comes from taking a square root). Let

$$\Gamma = \bigcup_{j=0}^{n+m-1} \Gamma_j \tag{2.3}$$

be the finite set of external rays landing at the critical orbit, and let $U \supset J$ be an open domain bounded by an equipotential curve of $f$. For $0 \leqslant j < \infty$, set $U_j = f^{-j}(U)$. Then $\bar{\Gamma}$ cuts $U = U_0$ into finitely many domains. This finite partition $\mathscr{P}_0 = \{\xi_i^0\}_{i=0}^{k_0}$ into open domains deserves to be called a *Markov partition* because it satisfies the following properties:

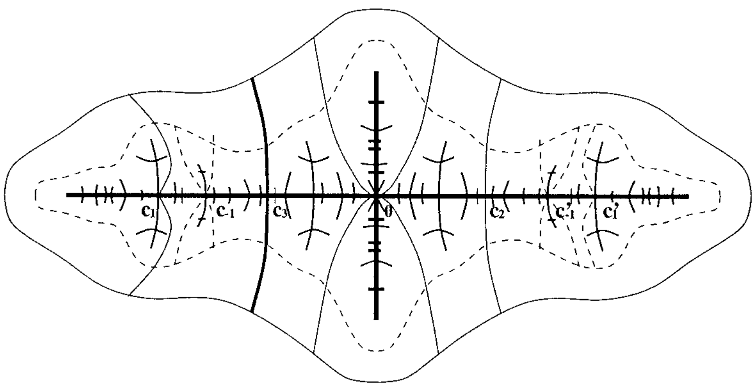
(1) The restriction of $f$ on each $\xi_i^0$ is injective.

(2) For each $\xi_i^0 \in \mathscr{P}_0$, the image $f(\xi_i^0 \cap U_1)$ is a union of domains in $\mathscr{P}_0$ (neglecting boundaries).

In Fig. 1, the Markov partition $\mathscr{P}_0$ is represented for the quadratic polynomial with $n = 3$ and $m = 1$, i.e., $f^3(0)$ is a repelling fixed point. (In this case, we have $q = 2$ external rays at $f^3(0) = c_3$.)

Pulling-back $\mathscr{P}_0$ via (both branches of) $f^{-1}$, we get a finer Markov partition $\mathscr{P}_1 = \mathscr{P}_0 \vee f^{-1}(\mathscr{P}_0)$, and construct inductively

$$\mathscr{P}_\ell = \mathscr{P}_{\ell-1} \vee f^{-\ell}(\mathscr{P}_0) = \{\xi_j^\ell\}_{j=1}^{k_\ell}, \tag{2.4}$$




0 --> $c_1$ --> $c_2$ --> $c_3$ --> $c_3$
$c_1'$ --> $c_2$
$c_{-1}$, $c_{-1}'$ --> 0
Bold solid curves are external rays landing at the fixed point $c_3$
Solid curves bound the initial Markov partition $P_0$
Dashed curves and solid curves bound our partition Q

Fig. 1. Partitions $\mathscr{P}_0$ and $\mathscr{Q}$ in case $n = 3$, $m = 1$.

where each $\xi_j^\ell$ is an open domain in $U_\ell$ such that $f^\ell\colon \xi_j^\ell \to \xi_j^0$ is a homeomorphism. Thus, we construct a sequence $\{\mathscr{P}_\ell\}_{\ell=0}^\infty$ of nested Markov partitions. Adapting to our slightly different partition the proof that the pieces of the Yoccoz puzzle containing a point $z$ shrink to that point for nonrenormalizable quadratic polynomials (see ref. 17, Theorem 5.7, Remarks 5.2 and 7.7(a)), we get that the maximal diameter $\max_j\{\mathrm{diam}(\xi_j^\ell)\}$ tends to zero as $\ell$ goes to infinity. This gives the following lemma, defining our initial partition:

**Lemma–Definition 2 (Partition $\mathscr{Q}$).** There is an integer $\ell_0 \geqslant 0$ such that $\bar{\xi} \cap f(C)$ contains at most one point for each $\xi = \xi_i^{\ell_0} \in \mathscr{P}_{\ell_0}$. Taking the smallest such integer $\ell_0$, define the partition $\mathscr{Q} = \mathscr{P}_{\ell_0}$.

In the case illustrated in Fig. 1 (where $\mathscr{Q}$ is also drawn), we have $\ell_0 = 1$.

Starting from $\mathscr{Q}$, we now construct a system of charts and dynamical conjugacies. It is convenient to introduce some further notation. We will say that a ray $\gamma$ is a *boundary ray* of a domain $\xi \in \mathscr{Q}$ if $\gamma \cap \bar{\xi}$ is non-empty. The landing point for each such ray is called a *vertex* point of $\xi$. By construction a vertex point $v$ is either in the critical orbit or a pre-image (up to $\ell_0$ iterates backwards) of the critical point. The closure of $\xi$ is the disjoint union of $\xi$, its boundary rays and its vertex points (plus a piece of an equipotential curve which is, however, not of interest here). An element $\xi \in \mathscr{Q}$ is called *non-singular* if its closure does not intersect the postcritical orbit $f(C)$, otherwise (i.e., if at least one of its vertices belongs to $f(C)$) it is called *singular*. In fact, when $\xi$ is singular, by definition of $\ell_0$, the intersection $\bar{\xi} \cap f(C)$ is a singleton $\{c_j\}$ for some $1 \leqslant j < n+m$; we then say that $\xi$ is *singular of index j,* and write $\mathrm{ind}(\xi) = j$. (For consistency we denote $\mathrm{ind}(\xi') = 0$ when $\xi'$ is not singular.) If $\xi, \eta \in \mathscr{Q}$ and $f(\eta) \supseteq \xi$, we write

$$f_{\eta\xi}^{-1}\colon \xi \to \eta$$

for the corresponding inverse map.

From now on, let us fix an external ray $\gamma \in \Gamma_n$. As in the construction of $\Gamma$ we may pull-back $\gamma = \gamma_n$ along the postcritical orbit taking suitable pre-images by $f$. All pull-backs are local diffeomorphisms, except for the last which produces two external rays $\gamma_{0\pm}$ at $c_0$ which cut the Riemann sphere into two topological disks $\mathscr{H}_\pm$. Choosing one of these, say $\mathscr{H}_+$ (we shall see below that this choice is immaterial, see e.g., Lemma 6), and denoting by $\gamma_1$ the image ray at $c_1$, we may define an inverse branch of $f_c$,

$$r(z) = \sqrt{z - c_1}, \tag{2.5}$$

which maps $\mathbb{C} \setminus \gamma_1$ conformally onto $\mathscr{H}_+$.

We shall now recursively define our dynamical coordinates, constructing a collection of (conformal) maps and open charts $\{(\psi_\xi, V_\xi) \mid \xi \in \mathcal{Q}\}$, with

$$\psi_\xi \colon \xi \to V_\xi \subset \mathbb{C}, \qquad \xi \in \mathcal{Q}.$$

**Definition 3 (Dynamical Charts).**

(1) If $\xi \in \mathcal{Q}$ is non-singular, then $V_\xi = \xi$ and $\psi_\xi(z) = z$ is the identity map.

(2) If $\xi$ is singular of index one, then $V_\xi = r(\xi)$, and $\psi_\xi(z) = r(z)$, i.e., $\psi$ is the restriction of the fixed inverse branch $r$.

(3) If $\xi$ is singular of index $j > 1$, there is a unique singular $\eta \in \mathcal{Q}$ which has index $j-1$ and such that $f(\eta) \supseteq \xi$. We define inductively $\psi_\xi$ (and $V_\xi$) by $\psi_\xi(z) := \psi_\eta \circ f^{-1}_{\eta\xi}(z)$ (this recursion is finite, with at most $n+m-1$ steps).

We may view the conjugacies $\psi_\xi$ either as holomorphic charts of our partition $\mathcal{Q}$ or as a map $\psi$ from a subset of the complex plane to a tower consisting in the disjoint union of finitely many open complex coordinate charts:

$$\psi \colon \bigcup_{\xi \in \mathcal{Q}} \xi \to \bigsqcup_{0 \leqslant j < n+m} \; \bigsqcup_{\eta' \in \mathcal{Q},\, \mathrm{ind}(\eta') = j} V_{\eta'}, \qquad \psi|_\xi = \psi_\xi.$$

Note that when $\xi \in \mathcal{Q}$ has index $j > 0$, we may also caracterise $V_\xi$ as the unique connected component in $\mathscr{H}_+$ with $0 = c_0 \in \overline{V_\xi}$ such that $f^j \colon V_\xi \to \xi$ is a conformal bijection (and $\psi_\xi$ is then the inverse map of $f^j$).

Finally, we construct a dynamical system on the tower, based on the inverse branches of $f$ in the new coordinates. For domains $\xi$ and $\eta$ in $\mathcal{Q}$ such that $f(\eta) \supset \xi$, we define a *conjugated inverse branch:*

$$g_{\eta\xi} = \psi_\eta \circ f^{-1}_{\eta\xi} \circ \psi_\xi^{-1} \colon V_\xi \to V_\eta.$$

## 3. PROPERTIES OF THE DYNAMICS IN THE NEW CHARTS

In this section, we list the properties of our preperiodic quadratic polynomial viewed in the charts constructed in Section 2. They will suffice to prove Theorem B in the next section. The first step is to extend the local inverse branches in the charts:

**Lemma 4 (Holomorphic Extension in Dynamical Charts).** There are open neighbourhoods $\Omega_\xi \supset \overline{V}_\xi$, for $\xi \in \mathcal{Q}$, such that each conjugated inverse branch $g_{\eta\xi} \colon V_\xi \to V_\eta$ extends holomorphically to $g_{\eta\xi} \colon \Omega_\xi \to \mathbb{C}$.

*Proof of Lemma 4.* When $\bar{\xi}$ does not intersect the postcritical orbit the conjugating map $\psi_\xi$ is the identity map and thus extends as a conformal bijection to a larger domain. Any inverse branch of $f$ also extends to a larger domain because the critical value is not on the boundary. The image of such an enlarged domain does not intersect the postcritical orbit either (if this domain is not taken too large) and $\psi_\eta$ extends holomorphically as well.

If $\xi$ is singular of index $j \geqslant 1$ but $\eta$ is not, then $\psi_\eta$ is the identity map and

$$g_{\eta\xi} = f_{\eta\xi}^{-1} \circ \psi_\xi^{-1} = f_{\eta\xi}^{-1} \circ f^j. \tag{3.1}$$

When $j > 1$, $f_{\eta\xi}^{-1}$ can be extended to a small neighbourhood of $\bar{\xi}$ (because $c_1 \notin \bar{\xi}$), and the map $g$ can be extended to a small neighbourhood of $\bar{V}_\xi$. (Note that in this case the origin is on the boundary of $V_\xi$ but that the conjugated map $g_{\eta\xi}(w) = \text{const} + w^2 + O(w^4)$ is not injective in a neighborhood of the origin). If $j = 1$, then $f_{\eta\xi}^{-1} = \pm r \mid \xi$. So $g_{\eta\xi}(z) = \pm z$.

When both $\xi$ and $\eta$ are singular there are two possibilities: First, $c_j \in \bar{\xi}$ and $c_{j-1} \in \bar{\eta}$ for some $1 < j < n+m$. From the definition of the $\psi$'s,

$$\psi_\xi = \psi_\eta \circ f_{\eta\xi}^{-1}.$$

Hence, $g_{\eta\xi}(z) = z$ is the identity map and can be extended holomorphically.

The other possibility (the most interesting one) is that $c_n \in \bar{\xi}$ and $c_{n+m-1} \in \bar{\eta}$. We need to map both $\xi$ and $\eta$ to domains containing the critical point on the boundary. We shall do so in two steps: First we conjugate to domains with the *critical value* on the boundary and thereafter to domains with the critical point on the boundary:

Let $\xi'$ be the (unique) singular domain of index $n$ such that $f^{m-1}(\xi') \supset \eta$, i.e., $f^m(\xi') \supset \xi$. The domains $\xi$ and $\xi'$ may be different (in fact, this happens precisely when $d > 1$ where $d$ was the divisor of $q$ in the Douady landing Theorem). Let $f_{\xi\xi'}^{-m}$ be the inverse of $f^m$: $\xi' \to \xi$. Then, by definition,

$$g_{\eta\xi} = \psi_\eta f_{\eta\xi}^{-1} \psi_\xi^{-1} = \psi_{\xi'} f_{\xi'\xi}^{-m} \psi_\xi^{-1}. \tag{3.2}$$

Let $\xi_1$ be the singular domain of index 1 such that $f^{n-1}(\xi_1) \supset \xi$. The corresponding inverse is $f_{\xi_1\xi}^{1-n}$: $\xi \to \xi_1$. We define similarly $\xi'_1$ and the associated maps (relative to $\xi'$).

Then for $z \in V_\xi$ we may express the conjugated map as follows:

$$g_{\eta\xi}(z) = r \circ f_{\xi'_1\xi'}^{1-n} f_{\xi'\xi}^{-m} f_{\xi\xi_1}^{n-1} \circ f(z) = r \circ F \circ f(z). \tag{3.3}$$

It is clear that $f^{-m}_{\xi'\xi}$ can be analytically extended to a small neighbourhood of $\bar{\xi}$ (since its linearisation around the fixed point $c_n$ is of the form $z \mapsto z/\lambda_f$). When starting from the critical value the associated iterates do not meet the critical point and therefore $f^{1-n}_{\xi_1 \xi}$ can also be analytically extended to a small neighbourhood of $\bar{\xi}$. These maps are all local diffeomorphisms. Hence, the middle composition $h$ has $c_1$ as a fixed point and in a neighborhood of this fixed point a local power series expansion,

$$F(w) = f^{1-n}_{\xi'_1 \xi'} f^{-m}_{\xi'\xi} f^{n-1}_{\xi\xi_1}(w) = c_1 + \lambda_f^{-1}(w - c_1) + O((w-c_1)^2). \tag{3.4}$$

By post- and precomposing with $r$ and $f$, respectively, we see that $g_{\eta\xi}$ can be extended to a small neighbourhood of $\bar{V}_\xi$ with the local power series expansion

$$g_{\eta\xi}(z) = \pm(\sqrt{\lambda_f})^{-1}\, z + O(z^2). \tag{3.5}$$

Note that when $d > 1$ the sign of the square root may depend on the choice of the ray $\gamma$ which was used to define the map $r$. This ambiguity will disappear in the dynamical determinant (where a $d$th iterate occur, see Lemma 6). ∎

Our next step is to obtain contraction in the sense of the Schwartz lemma (we will thus be in a position to apply standard results of Ruelle on Fredholm determinants for holomorphic contractions). The basic idea in the proof is fairly standard in complex dynamics (using "thickened" puzzle pieces), but some complications arise from the conjugacies.

**Lemma 5 (Contraction in Dynamical Charts).** Up to taking slightly smaller domains $\Omega_\xi \supset \bar{V}_\xi$ than in Lemma 4, we have

$$g_{\eta\xi}(\bar{\Omega}_\xi) \subset \Omega_\eta$$

for each conjugated inverse branch $g_{\eta\xi}\colon \Omega_\xi \to \Omega_\eta$, and $\xi, \eta \in \mathcal{Q}$.

*Proof of Lemma 5.* There are two steps. First we will cover the postcritical orbit and up to $\ell_0$ pre-images thereof by a suitable collection of small balls. These will provide domains for local contractions. We will then use slightly "thickened" external rays to fill in and obtain global contractions (see ref. 13, pp. 213–214 for a similar construction).

Consider a small open ball $B(c_n, \delta)$ of radius $\delta > 0$ centered at the point, $c_n$. Since $c_n = c_{n+m}$ is a repelling periodic point of $f^m$, there are $\delta^* > 0$ and $\rho > 1$ such that for $0 < \delta < \delta^*$, we have $f^m(B(c_n, \delta)) \supset \overline{B(c_n, \delta\rho^m)}$. When $v \in \bar{\xi}$ is a vertex point there is $j \in \{-\ell_0, \ldots, n+m-1\}$

(called the index of the vertex point) such that $f^{j-n}(c_n)=v$. Let $B_v$ be the connected component of $f^{j-n}(B(c_n,\delta\rho^{n-j}))$ which contains the vertex point, $v$ (when $j\geqslant n$ we consider the image, while for $j<n$ we take the pre-image). Since $\ell_0$ is finite we may take $\delta$ small enough so that the collection of balls, $B_v$, $v$ a vertex, is disjoint.

Consider a transition $\eta\to\xi$ between two domains. Let $v\in\bar{\xi}$ be a vertex point of $\xi$ and let $w=f_{\eta\xi}^{-1}(v)\in\bar{\eta}$. There are three possibilities: either, case I, $w$ is itself a vertex point of $\eta$ of index one smaller than the index of $v$, case II, $v=c_n$ and $w=c_{n+m-1}$, or, case III, $w$ is not a vertex point but belongs to the open domain $\eta$. In case I, the fact that $\rho>1$ ensures that $f_{\eta\xi}^{-1}(\overline{B_v})\subset B_w$. In case II, the choice of $\rho$ and $\delta$ was such that, $f_{\eta\xi}^{-1}(\overline{B_v})\subset f_{\eta\xi}^{m-1}(\overline{B(c_n,\delta\rho^{-m})})\subset B_w$. Finally, in case III, possibly by making $\delta$ smaller, we may ensure that $f_{\eta\xi}^{-1}(\overline{B_v})\subset\eta$.

In summary, with suitable choices of $\delta$ and $\rho$ we have that $f_{\eta\xi}^{-1}(\overline{B_v})\subset\eta\cup\bigcup_w B_w$, the union being taken over vertices of $\eta$. This ends the first step in the construction.

Now, let $\gamma_\theta$ be an external ray bounding a domain $\xi\in\mathcal{Q}$, where $\theta$ is its angle, and let $v$ be its landing point (which is a vertex of $\xi$). We may choose $\epsilon>0$ small enough so that every external ray (independently of whether or not it lands) with angle in $(\theta-\epsilon,\theta+\epsilon)$ will intersect the ball $\overline{B_v}$. As the number of vertices is finite, $\epsilon$ can be chosen uniformly in $v$. Let $\gamma'_\zeta$ denote the part of the ray of angle $\zeta$ between its entry in the equipotential neighbourhood $U_{\ell_0}$ and its first intersection with $\overline{B_v}$. Finally, let $\gamma'_{\theta\pm\epsilon}$ denote the "thickened ray" which is the union of all $\gamma'_\zeta$ for $\zeta\in(\theta-\epsilon,\theta+\epsilon)$.

Consider now a pre-image $\gamma_\zeta$ of $\gamma_\theta$, and let $w$ be its landing point. The pre-image is either bounding some domain $\eta\in\mathcal{Q}$, or it is inside such a domain. In the first case, the pre-image possesses a neighbourhood $\gamma'_{\zeta\pm\epsilon}$ of rays, the point $w$ is a vertex of $\eta$, and by construction one has (chosing for the inverse the connected component intersecting $\gamma'_\zeta$) $f_{\eta\xi}^{-1}(\overline{\gamma'_{\theta\pm\epsilon}})\subset\gamma'_{\zeta\pm\epsilon}\cup B_w$, essentially because a square root halves angle-differences. For the second case, we may choose a smaller $\epsilon$ so that (uniformly in these rays), $f_{\eta\xi}^{-1}(\overline{\gamma'_{\theta\pm\epsilon}})\subset\eta$.

When the domain $\xi$ is non-singular, we simply define $\Omega_\xi=V_\xi\bigcup_v B_v\bigcup_i\gamma'_{\theta_i\pm\epsilon}$ the unions being over vertices and bounding rays of the domain. If the domain is singular, say of index $j>0$, then $\psi_\xi$ maps $\xi$ to the domain $V_\xi$ containing the origin at the boundary. If $\gamma_\theta$ is a ray which bounds $\xi$ then it has one or two pre-image(s) which bound(s) $V_\xi$ (i.e., intersect(s) the closure of $V_\xi$). To each such ray pre-image we associate a corresponding pre-image (the connected component containing the ray), $\gamma'_{\theta'\pm\epsilon}$, of the thickened ray $\gamma'_{\theta\pm\epsilon}$. Finally, to each vertex $v$, we associate a connected component $B'_v$ of the corresponding ball $B_v$. We let $\Omega_\xi=V_\xi\bigcup_{\theta'}\gamma'_{\theta'\pm\epsilon}\bigcup_v B'_v$ and one verifies that by construction each $g_{\eta\xi}(\bar{\Omega}_\xi)\subset\Omega_\eta$.

Some remarks may clarify the procedure in the singular case: First note that if only one ray, say $\gamma_\theta$, lands at $c_n$, then both pre-images of $\gamma'_{\theta\pm\epsilon}$ will bound the domain $V_\xi$ and hence both should be included in $\Omega_\xi$. Second, each $V_\xi$ is an open subset of the "half-plane" $\mathscr{H}_+$ (since it is in the image of $r$). The extended domain $\Omega_\xi$, however, is a neighbourhood of the closure of $V_\xi$, in particular it contains a neighbourhood of the origin. Recall that if $\eta$ was non-singular then $g_{\eta\xi}$ behaves as a quadratic map around the origin. This illustrates a rather peculiar effect of our dynamical conjugation, namely that on the extended domains, the (noninjective) map $g_{\eta\xi}$ is a strict contraction but not the inverse of an expanding map. (This reflects the fact that our construction is just another description of the orbifold metric, see comments after the proof of Theorem B.) ▮

It remains to understand the relation between the periodic orbits and multipliers of $f: \mathbb{C} \to \mathbb{C}$ and those of the system of contractions

$$g = \{g_{\xi\eta}\}: \Omega \to \Omega, \qquad \Omega = \bigsqcup_{\xi \in \mathscr{Q}} \Omega_\xi.$$

(Slightly abusing terminology, we continue to call vertex of an $\Omega_\xi$ a vertex of $\xi$.) In Lemma 6 we shall see that, except for the postcritical repelling point, nothing changes. For this repelling periodic point, however, the period is multiplied by $d \geqslant 1$, and the multiplier must be replaced by an appropriate square root of $\lambda_f^d$. Let us first define this square root:

**Dynamical Choice of $\sqrt{\lambda_f^d}$.** The external ray $\gamma$ landing at $c_n$ that we used to construct the partition $\mathscr{Q}$ may be parametrised as a real-analytic map $t \in \mathbb{R} \mapsto \gamma(t) \in \mathbb{C} - \{c_n\}$, where $\lim_{t \to \infty} \gamma(t) = \infty$ and $\lim_{t \to -\infty} \gamma(t) = c_n$. (We shall not make use of the fact that $\gamma$ is real-analytic, we only need the continuity property.) The ray is invariant under the $md$th iterate of the map $f$, and it is convenient to choose the parametrisation (through a simple expression involving the Boettcher conjugacy $\phi$) so that for all $t \in \mathbb{R}$,

$$f^{md}(\gamma(t)) = \gamma(t + md). \tag{3.6}$$

Since $\gamma$ is a continuous curve and does not contain $c_n$, we may write

$$\gamma(t) = c_n + \exp(\alpha(t)), \tag{3.7}$$

where $\alpha$ is unique up to multiples of $2\pi i$. By linearizing the invariance relation (3.6) for $\gamma$, we see that (note here the extra $d$th power)

$$\lim_{t \to -\infty} \exp(\alpha(t) - \alpha(t - md)) = \lambda_f^d. \tag{3.8}$$

Note that from (3.6), (3.7) and the fact that $c_n$ is a strictly repelling periodic point, the convergence in (3.8) takes place at an exponential rate (because $|\exp(\alpha(t))|$, which describes the correction to the right-hand-side of (3.8), tends exponentially fast to zero as $t \to \infty$). The following limit thus also exists and gives a logarithm for $\lambda_f^d$:

$$\lim_{t \to -\infty} \alpha(t) - \alpha(t - md). \tag{3.9}$$

Recall that the last two external rays $\gamma_{0\pm}$ at $c_0$ obtained by pulling back $\gamma$ in the construction of our partition arose from taking a square root. We may thus write

$$\gamma_{0\pm}(t) = f^{-n}(\gamma(t)) = c_0 \pm \exp(\alpha_0(t)). \tag{3.10}$$

Here, $\alpha_0$ is unique up to multiples of $\pi i$, whence the plus-minus. Through linearisation we see that (the constant is a logarithm of the derivative along the postcritical orbit)

$$\lim_{t \to -\infty} \alpha_0(t) - \alpha(t)/2 = \text{const.} \tag{3.11}$$

The pull-back along the postcritical orbit is holomorphic with a non-zero derivative. It follows that the convergence in (3.11) also takes place at an exponential rate with a correction of the order of $|\exp(\alpha(t))|$. This allows us finally to define a square root of the multiplier (morally, we are defining a logarithm of the square root, see (3.9)) :

$$\sqrt{\lambda_f^d} \equiv \lim_{t \to -\infty} e^{\alpha_0(t) - \alpha_0(t - md)} = \lim_{t \to -\infty} e^{(\alpha(t) - \alpha(t - md))/2}. \tag{3.12}$$

It is interesting to note that this definition does not depend on the choice of the parametrisations $\alpha$ and $\alpha_0$ (which is clear, including the fact that $+\gamma_0$ and $-\gamma_0$ lead to the same choice), nor of the initial ray $\gamma$ landing at $c_n$. To see the latter we will use a simple homotopy argument: Let $\tilde{\gamma}$ be another ray landing at $c_n$. It does not intersect $\gamma$ and a logarithm of the parametrisation, $\tilde{\alpha}$, can thus not intersect the family of curves $\alpha + i\mathbb{Z}$. The same must be true for the pulled-back rays and this forces the limit $\lim_{t \to -\infty} \lim_{k \to \infty} (\tilde{\alpha}_0(t) - \tilde{\alpha}_0(t - mdk))/k$ to be independent of the various choices. But this limit yields the unique (relative to the family of rays) logarithm of $\sqrt{\lambda_f^d}$. Taking exponentials we obtain the wanted conclusion.

We can now state the last ingredient to prove Theorem B:

**Lemma 6 (Periodic Orbits of *f* and *g*).** There is an isomorphism between the set of periodic points for $f$ which are not in the postcritical periodic orbit and the set of periodic points of $g: \Omega \to \Omega$ which are not vertices of domains. The isomorphism preserves the period and the multiplier of these periodic points.

Let $m \cdot d$ be as in the Douady Landing Theorem. To each $c_j$, with $n \leqslant j < n+m$ in the postcritical periodic orbit, there corresponds $q$ periodic points of $g$ which are vertices of domains. Each such point $z$ has period $m \cdot d$ and multiplier $1/\sqrt{\lambda_f^d}$ (as defined in (3.12)). There are no other periodic points for $g$.

*Proof of Lemma 6.* Consider a point $z \in \Omega_\xi$ for some $\xi \in \mathcal{Q}$. This point is periodic of period $k \geqslant 1$ for $g$ precisely when there is a symbol sequence $\xi_0, \ldots, \xi_{k-1}$, with $\xi = \xi_0$ and such that there are transitions $\xi_j \to \xi_{j+1 \bmod k}$ for each $j = 0, \ldots, k-1$ (we say such a cyclic symbol sequence is admissible). In that case we have $g_{\xi_0 \xi_1} \circ \cdots \circ g_{\xi_{k-1} \xi_0}(z) = z$ and furthermore, either $z \in V_\xi$ (i.e., in the chart of the open domain of the original partition) or $z$ is (a lift of a point) in the postcritical periodic orbit. To see this note first that contraction of the family $g$ implies that each admissible cyclic symbol sequence yields a unique fixed point $z \in \Omega_{\xi_0} = \Omega_\xi$ following that sequence. Now, $g_{\eta\xi}: \Omega_\xi \to \Omega_\eta$ in fact maps $V_\xi$ to $V_\eta$, and it follows that the fixed point must be in the closure $\bar{V}_\xi$ of the domain in question. If $z$ is not in the interior it must be on the boundary. Clearly it can not be on the piece of equipotential curve and no ray $\gamma_\theta$ which bounds the domain can contain any periodic points. Hence $z$ must be a vertex of the domain and only vertices which are lifts of points in the postcritical periodic orbit can be periodic.

Since on the open domains $f$ and $g$ are conjugated by diffeomorphisms, we have already proved the first claim.

We now prove the remaining statements. Since there are $q$ rays landing at $c_n$, there are $q$ domains in $\mathcal{Q}$ having $c_n$ as a vertex. The same holds for all points in the postcritical orbit, whence the (over)counting of the corresponding periodic vertices for $g$. The (minimal) period $m \cdot d$ of the rays gives the same minimal period of domains under the mapping $g$, proving our claim on the period of the vertices. Finally to get the multiplier of the return map of a periodic vertex $z$, let $\xi$ be a domain having $z = c_n$ as a vertex. Take any $w \in \Omega_\xi - \{c_0 = 0\}$. Then the multiplier will be the limit of $(g^{mdk}(w) - c_0)/(g^{md(k-1)}(w) - c_0)$ as $k \to \infty$. If here we let $w$ be a point on the landing ray $\gamma_0^+$ then it is easy to check that this limit coincides with the one over the limit in (3.12) for $\sqrt{\lambda_f^d}$. (Recall that if we took $\gamma_0^-$ or another ray we would get the same limit, by the observations made above. Note also that the choice between $\mathscr{H}^+$ and $\mathscr{H}^-$ does not intervene here.) ▮

## 4. DYNAMICAL DETERMINANTS—PROOFS OF THE THEOREMS

We first give the simple proof of the generalised Levin–Sodin–Yuditskii[(6)] formula:

*Proof of Theorem A.* One views the second equality in (1.3) as an identity between formal power series, i.e., between the critical orbit power series and the exponential of the sum over fixed points. The coefficient of $z^k$ is thus the same on both sides when $-c$ is real and large enough. We will show that each such coefficient $a_k(c)$ of the left-hand-side (involving the periodic points) extends as a meromorphic function in $c$ to the whole plane. Uniqueness of meromorphic extensions will then give Theorem A.

Since the coefficient $a_k(c)$ of $z^k$ on the left-hand-side is a sum and product of terms which only involve sums over fixed points of order not greater than $k$, it suffices to show that each term of the type

$$\sum_{w \in \mathrm{Fix}\, f_c^k} \frac{1}{(f_c^k)'(w)((f_c^k)'(w)-1)} \tag{4.1}$$

defines a (single-valued) meromorphic function of $c$. Assume now that for each $k$, the zeroes $w(c)$ of

$$F(w, c) = f_c^k(w) - w$$

are isolated as a function of $c$. (This will be proved in Sublemma 7 below.) Then the sum over Fix $f^k$ has only finitely many singular points, solutions to algebraic equations. Take a loop in the complex plane which avoids these singularities. Traversing the loop yields a permutation of the fixed points. Hence the sum is unchanged, and the function is single-valued as we wanted to show. ▮

**Sublemma 7.** Let $F(w, c)$, $G(w, c)$, and $H(w, c)$ be polynomials in $w$ and $c$ such that:

1. $F(w, c) - w^N$ is a polynomial in $w$ and $c$ of degree at most $N-1$ in $w$.

2. There is $\hat{c}$ such that $H(w, \hat{c}) \neq 0$ at all $w$ for which $F(w, \hat{c}) = 0$ (this guarantees that the function $R(c)$ below is well-defined on a small disc).

Then the function

$$R(c) = \sum_{w:\, F(w,c)=0} \frac{G(w,c)}{H(w,c)} \tag{4.2}$$

is rational.

In our application, we take $F(w, c) = f_c^k(w) - w$, $G(w, c) = 1$, $H(w, c) = (f_c^j)'(w)((f_c^j)'(w) - 1)$, and $N = 2k$. We note (although we shall not need this) that one can extend the above result to more general holomorphic or even meromorphic functions.

*Proof of Sublemma 7.* Let $w_1,\dots, w_N$ be $N$ complex variables, and let $\sigma_1,\dots, \sigma_N$ denote the elementary symmetric functions

$$\sigma_1 = w_1 + \cdots + w_N, \qquad \sigma_2 = \sum_{i<j} w_i w_j, \dots, \qquad \sigma_N = w_1 \cdots w_N.$$

For fixed $c$, let $\{w_1(c),\dots, w_N(c)\}$ denote the zero-set $\{w\colon F(w, c) = 0\}$ (counted with multiplicity, we use the degree claim in (1) here). We have

$$F(w,c) = (w - w_1(c)) \cdots (w - w_N(c)) = w^N - \sigma_1(c)\, w^{N-1} \pm \cdots \sigma_N(c), \tag{4.3}$$

where $\sigma_1(c),\dots, \sigma_N(c)$ are the elementary polynomials evaluated in the roots $w_i(c)$. Each $\sigma_i(c)$ is a polynomial function of $c$ because we assumed $F(w, c)$ to be polynomial in $c$.

We can now conclude: The function $H^*(c) = \prod_{w = w_i(c)} H(w, c)$ is a symmetric polynomial in the roots $w_i(c)$. The fundamental theorem of symmetric functions (see, e.g., ref. 23) asserts that it is a polynomial in the elementary functions evaluated at the roots, i.e., $H^*(c) = \pi_H(\sigma_1(c),\dots, \sigma_N(c))$, for some polynomial $\pi_H$. By the observation in the previous paragraph, $H^*(c)$ is thus a polynomial in $c$.

Finally,

$$R(c)\, H^*(c) = \sum_{u = w_i(c)} G(u,c) \prod_{v = w_j(c):\, j \neq i} H(v,c) \tag{4.4}$$

is also a symmetric polynomial in the roots hence a polynomial in $c$ (not identically zero by the assumptions on $F$ and $H$). It follows that $R(c)$ is rational. ∎

*Proof of Theorem B.* Write $\hat{h}_\xi = h \circ \psi_\xi^{-1}$ on $V_\xi$, and note that $\hat{h}_x i$ extends holomorphically to $\Omega_x i$. By Lemmas 4 and 5, the Markov system of holomorphic contractions $g: \Omega \to \Omega$ and (for any integer $\beta$) the holomorphic weight $\hat{h} \cdot (g')^\beta$ on $\Omega$ satisfy the assumptions of Theorem 1 in ref. 5. In particular, the transfer operator $\hat{\mathscr{L}}_\beta$ acting on the Banach space $\hat{\mathscr{B}}$ of holomorphic functions on $\Omega$ which admit a continuous extension to the closure of $\Omega$ (endowed with the supremum norm) through

$$\hat{\mathscr{L}}_\beta \hat{\varphi}(\hat{w}) = \sum_{\text{admissible } \eta} \hat{h}(\hat{w})(g'_{\eta\xi}(\hat{w}))^\beta \cdot \hat{\varphi}(g_{\eta\xi}(\hat{w})), \qquad \hat{w} \in \Omega_\xi. \tag{4.5}$$

is nuclear of order zero. (Indeed, as observed in ref. 5, p. 234, lines 1–10, case $k=0$, this operator is bounded from the nuclear space $\hat{\mathscr{H}}$ of holomorphic functions on $\Omega$ to the Banach space $\hat{\mathscr{B}}$, and the inclusion of $\hat{\mathscr{B}}$ in $\hat{\mathscr{H}}$ is continuous, so that we may apply [ref. 24, I: p. 84, and II: p. 9, Cor. p. 56, Cor. 4 p. 39, Cor. 2 p. 61].) Therefore, just as in the proof of Theorem 1 in ref. 5, we may apply Cor. 4 p.18 in [ref. 24, II] to see that $\hat{\mathscr{L}}_\beta$ has a trace and a (Fredholm–Grothendieck) determinant

$$\text{tr } \hat{\mathscr{L}}_\beta = \sum_i u_i, \qquad \det(1 - z\hat{\mathscr{L}}_\beta) = \prod_i (1 - zu_i),$$

where the $u_i$ are the eigenvalues of $\hat{\mathscr{L}}_\beta$ repeated according to multiplicity. Furthermore, $\det(1 - z\hat{\mathscr{L}}_\beta)$ is an entire function of order zero ([ref. 24, II, Thm. 4, p. 16]). Using Cauchy kernels, Ruelle [ref. 5, line 17 of p. 235] proved that the trace is given by

$$\text{tr } \hat{\mathscr{L}}_\beta = \sum_{\hat{w} \in \text{Fix } g} \hat{h}(\hat{w}) \cdot (g'(\hat{w}))^\beta \frac{1}{1 - g'(\hat{w})}.$$

By Grothendieck's Fredholm theory, the trace yields the determinant through

$$\det(1 - z\hat{\mathscr{L}}_\beta) = \exp - \sum_{k=1}^{\infty} \frac{z^k}{k} \text{tr } \hat{\mathscr{L}}_\beta^k.$$

Note that, clearly,

$$\text{tr } \hat{\mathscr{L}}_\beta^k = \sum_{\hat{w} \in \text{Fix } g^k} \prod_{\ell=0}^{k-1} \hat{h}(g^\ell(\hat{w})) \cdot ((g^k)'(\hat{w}))^\beta \frac{1}{1 - (g^k)'(\hat{w})}.$$

By Lemma 6, the only difference between the determinant $\hat{d}_\beta(z) = \det(1 - z\hat{\mathscr{L}}_\beta)$ of $\hat{\mathscr{L}}_\beta$ and the formal determinant $d_\beta(z)$ is the fact that the postcritical point of period $m$ is (over)counted $q \geqslant 1$ times in the tower, its period is multiplied by $d$, and its multiplier is $\sqrt{\lambda_f^d}$ instead of $\lambda_f^d$. Since

$$\exp\left[-\sum_{k\geqslant 1}\frac{z^{kmd}}{km}\frac{qm}{(h_m^d)^k(\sqrt{\lambda_f^d})^{k\beta}}\frac{1}{1-\frac{1}{(\sqrt{\lambda_f^d})^k}}+\sum_{k\geqslant 1}\frac{z^{km}}{km}\frac{m}{h_m^k\lambda_f^{k\beta}}\frac{1}{1-\frac{1}{\lambda_f^k}}\right]$$
$$=\prod_{j\geqslant 0}\left(1-\frac{z^{dm}}{h_m^d\sqrt{\lambda_f^d}^{\beta}}\frac{1}{\sqrt{\lambda_f^d}^{j}}\right)^q\cdot\prod_{j\geqslant 0}\left(1-\frac{z^{dm}}{h_m\lambda_f^\beta}\frac{1}{\lambda_f^j}\right)^{-1},$$

the resulting formula is as described in Theorem B.

Recall that the conjugacies $\psi_\xi\colon \xi\to V_\xi$ are holomorphic, and that $g_{\eta\xi}\circ\psi_\xi=\psi_\eta\circ f_{\eta\xi}^{-1}$ on $\xi$, so that $[(g'_{\eta\xi}\circ\psi_\xi)\cdot\psi'_\xi]^\beta=[(\psi'_\eta\circ f_{\eta\xi}^{-1})\cdot(f_{\eta\xi}^{-1})']^\beta$ there. We may use this conjugacy to define a Banach subspace $\mathscr{B}=\mathscr{B}_\beta$ of the set of functions $\varphi$ which are holomorphic on the (disjoint) union of the open domains $\xi\in\mathscr{Q}$. For this, we consider the continuous embedding of $\hat{\mathscr{B}}$ through

$$\varphi|_\xi=(\psi'_\xi)^\beta\cdot\hat{\varphi}_{|V_\xi}\circ\psi_\xi.\qquad \xi\in\mathscr{Q}, \tag{4.6}$$

Uniqueness of analytic continuations implies that this map is injective (hence we get an embedding), and we may therefore define the norm of $\varphi$ to be the corresponding (sup-)norm of $\hat{\varphi}\in\hat{\mathscr{B}}$ (see ref. 25 for a similar construction). Note that $\varphi\in\mathscr{B}_\beta$ is such that $((\psi_\xi^{-1})')^\beta\cdot(\varphi_{|\xi}\circ\psi_\xi^{-1})$ extends to a bounded holomorphic function on $\Omega_\xi$, thus allowing a specific type of singularities of $\varphi$ along the postcritical orbit. The weights in the above conjugacy have been chosen so that $\hat{\mathscr{L}}_\beta$ is conjugated to $\mathscr{L}_\beta$ through the above isometry, proving our last claim. ∎

To analyse subhyperbolic quadratic polynomials, Douady–Hubbard[11] construct a covering $\tilde{\Omega}$ over a neighbourhood of the Julia set, ramified above the postcritical orbit and endow it with a metric which makes the lifted polynomial $\tilde{f}$ uniformly expanding (see also [ref. 26, Section 3.2 and Appendix A] for a description of this orbifold metric). Constructing a Markov partition $\tilde{\mathscr{Q}}$ of $\tilde{\Omega}$ with ideas similar to those of the present paper, one could define a Banach space $\tilde{\mathscr{B}}$ of holomorphic bounded functions on $\tilde{\Omega}$ (which would be just another way to see $\hat{\mathscr{B}}$) such that the transfer operator associated to $\tilde{f}$ and the weight $(\tilde{f}')^{-\beta}$ (with the modifications inherent to the partition $\tilde{\mathscr{Q}}$) is nuclear. We leave out the details in this approach.

***Proof of Theorem B′.*** The Douady–Hubbard Theorem says that a quadratic-like map with connected filled-in Julia set is conjugated to a unique quadratic polynomial by a quasiconformal homeomorphism (see, e.g., ref. 13, pp. 192–194). The filled-in Julia set of our subhyperbolic quadratic-like map is connected, so that this quasiconformal homeomorphism allows us to construct a Markov partition, just like in Section 2, using the (quasiconformal) images of the external rays of the conjugated quadratic polynomial. The rest of the argument follows exactly as in the case of a polynomial (see Section 3 and the proof of Theorem B). ∎

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